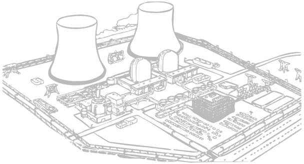

Estimating Value at Risk

Eric Marsden

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Do you know how risky your bank is?

Learning objectives

1. Understand measures of financial risk, including *Value at Risk*
2. Understand the impact of correlated risks
3. Know how to use copulas to sample from a multivariate probability distribution, including correlation

The information presented here is pedagogical in nature and does not constitute investment advice!

Methods used here can also be applied to model natural hazards

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Warmup. Before reading this material, we suggest you consult the following associated slides:

- Modelling correlations using Python
- Statistical modelling with Python

Available from `risk-engineering.org`

Risk in finance

> *There are 10^{11} stars in the galaxy. That used to be a huge number. But it's only a hundred billion. It's less than the national deficit! We used to call them* ***astronomical*** *numbers. Now we should call them* ***economical*** *numbers.*
>
> *— Richard Feynman*

Terminology in finance

Names of some instruments used in finance:

- A **bond** issued by a company or a government is just a loan
  - bond buyer lends money to bond issuer
  - issuer will return money plus some interest when the bond matures
- A **stock** gives you (a small fraction of) ownership in a "listed company"
  - a stock has a price, and can be bought and sold on the stock market
- A **future** is a promise to do a transaction at a later date
  - refers to some "underlying" product which will be bought or sold at a later time
  - example: farmer can sell her crop before harvest, at a fixed price
  - way of transferring risk: farmer protected from risk of price drop, but also from possibility of unexpected profit if price increases

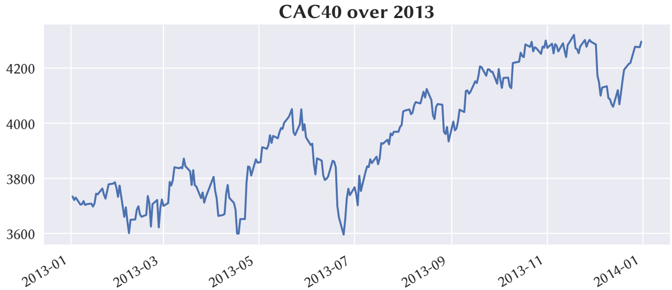
Risk in finance

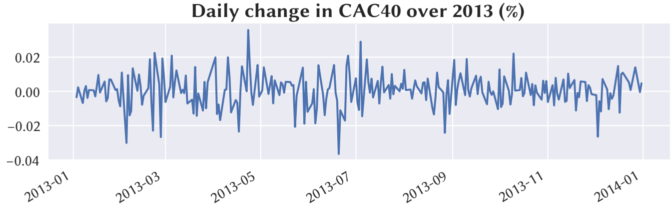
- Possible definitions:
  - "any event or action that may adversely affect an organization's ability to achieve its objectives and execute its strategies"
  - "the quantifiable likelihood of loss or less-than-expected returns"

- Main categories:
  - **market risk**: change in the value of a financial position due to changes in the value of the underlying components on which that position depends, such as stock and bond prices, exchange rates, commodity prices
  - **credit risk**: not receiving promised repayments on outstanding investments such as loans and bonds, because of the "default" of the borrower
  - **operational risk**: losses resulting from inadequate or failed internal processes, people and systems, or from external events
  - **underwriting risk**: inherent in insurance policies sold, due to changing patterns in natural hazards, in demographic tables (life insurance), in consumer behaviour, and due to systemic risks

Source: *Quantitative Risk Management: Concepts, Techniques and Tools*, A. J. McNeil, R. Frey, P. Embrechts

Stock market returns





Say we have a stock portfolio. How risky is our investment?

We want to model the likelihood that our stock portfolio loses money.

Value at Risk

- Objective: produce a **single number** to summarize my **exposure to market risk**
  - naïve approach: *How much could I lose in the "worst" scenario?*
  - that's a bad question: you could lose everything
- A more informative question:
  - *"What is the loss level that we are X% confident will not be exceeded in N business days?"*
- "5-day $VaR_{0.9}$ = 10 M€" tells us:
  - *I am 90% sure I won't lose more than 10 M€ in the next 5 trading days*
  - *There is 90% chance that my loss will be smaller than 10 M€ in the next 5 days*
  - *There is 10% chance that my loss will be larger than 10 M€ in the next 5 days*
- What it does not tell us:
  - *How much could I lose in those 10% of scenarios?*

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Value at Risk

Value at risk

A measure of market risk, which uses the statistical analysis of historical market trends and volatilities to estimate the likelihood that a given portfolio's losses (L) will exceed a certain amount l.

$$\mathrm{VaR}_\alpha(L) = \inf\{l \in \mathbb{R} : \Pr(L > l) \leq 1 - \alpha\}$$

where L is the loss of the portfolio and $\alpha \in [0, 1]$ is the confidence level.

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Value at Risk

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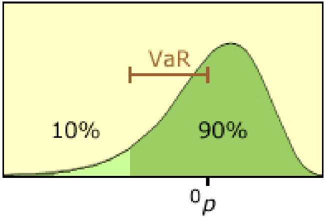
If a portfolio of stocks has a one-day 10% VaR of 1 M€, there is a 10% probability that the portfolio will decline in value by more than 1 M€ over the next day, assuming that markets are normal.



Applications of VaR

- **Risk management**: how much financial risk am I exposed to?
  - Provides a structured methodology for critically thinking about risk, and consolidating risk across an organization
  - VaR can be applied to individual stocks, portfolios of stocks, hedge funds, *etc.*
- **Risk limit setting** (internal controls or regulator imposed)
  - Basel II Accord (financial regulation) attempts to ensure that a bank has adequate capital for the risk that the bank exposes itself to through its lending and investment practices
  - VaR is often used as a measure of market risk
  - Provides a single number which is easy to understand by non-specialists

Limitations of VaR

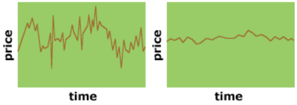
- Typical VaR estimation methods assume "normal" market conditions
- They do not attempt to assess the potential impact of "**black swan**" **events**
  - outlier events that carry an extreme impact
  - example: effects of cascading failure in the banking industry, such as the 2008 subprime mortgage crisis
- More information: see the slides on *Black swans* at `risk-engineering.org`

Alternatives to VaR

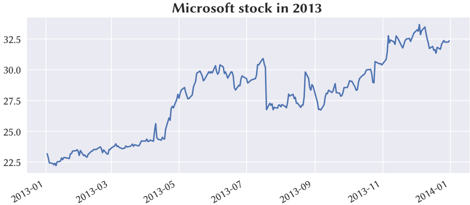
- VaR is a *frequency* measure, not a *severity* measure
  - it's a *threshold*, not an expectation of the amount lost
- Related risk measure: Expected Shortfall, the average loss for losses larger than the VaR
  - expected shortfall at $q\%$ level is the expected return in the worst $q\%$ of cases
  - also called *conditional value at risk* (CVaR) and *expected tail loss*
- Note that
  - ES_q increases as q increases
  - ES_q is always greater than VaR_q at the same q level (for the same portfolio)
- Unlike VaR, expected shortfall is a *coherent* risk measure
  - a risk measure $\mathcal{R}$ is *subadditive* if $\mathcal{R}(X+Y) \leq \mathcal{R}(X) + \mathcal{R}(Y)$
  - the risk of two portfolios combined cannot exceed the risk of the two separate portfolios added together (diversification does not increase risk)

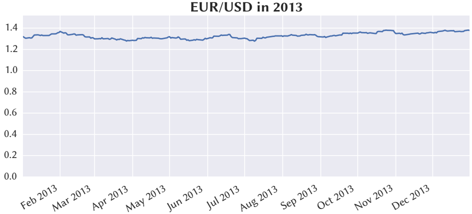
Estimating VaR

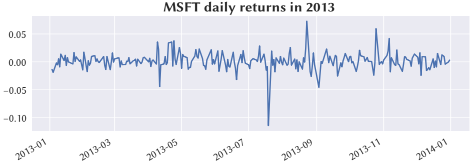
- Estimation is difficult because we are dealing with **rare events** whose probability distribution is unknown
- Three main methods are used to estimate VaR:
  1. historical bootstrap method
  2. variance-covariance method
  3. Monte Carlo simulation
- All are based on estimating **volatility**
- Applications of the *constant expected return* model which is widely used in finance
  - assumption: an asset's return over time is independent and identically normally distributed with a constant (time invariant) mean and variance

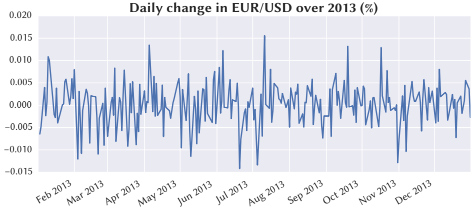


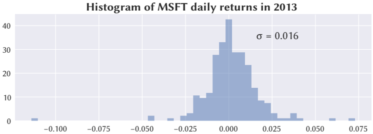
Understanding volatility

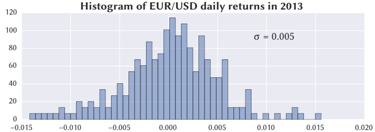










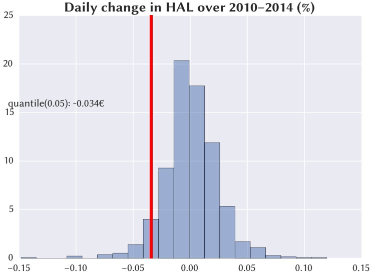


high volatility

low volatility

Historical bootstrap method

- Hypothesis: history is representative of future activity
- Method: calculate **empirical quantiles** from a histogram of daily returns
- 0.05 empirical quantile of daily returns is at -0.034:
  - with 95% confidence, our worst daily loss will not exceed 3.4%
  - 1 M€ investment: one-day 5% VaR is 0.034 × 1 M€ = 34 k€
  - (note: the 0.05 quantile is the 5th percentile)
- 0.01 empirical quantile of daily returns is at -0.062:
  - with 99% confidence, our worst daily loss will not exceed 6.2%
  - 1 M€ investment: one-day 1% VaR is 0.062 × 1 M€ = 62 k€



Download this content as a Python notebook at `risk-engineering.org`

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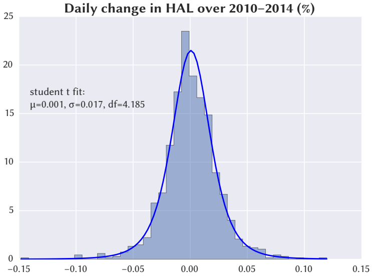
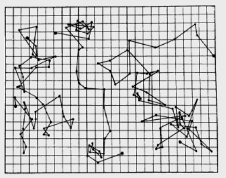
Variance-covariance method

- Hypothesis: daily returns are normally distributed
- Method: analytic quantiles by **curve fitting to historical data**
  - here: Student's t distribution
- 0.05 analytic quantile is at -0.0384
  - with 95% confidence, our worst daily loss will not exceed 3.84%
  - 1 M€ investment: one-day 5% VaR is 0.0384 × 1 M€ = 38 k€
- 0.01 analytic quantile is at -0.0546
  - with 99% confidence, our worst daily loss will not exceed 5.46%
  - 1 M€ investment: one-day 1% VaR is 0.0546 × 1 M€ = 54 k€



Monte Carlo simulation



- ▷ Method:
  1. run many "trials" with random market conditions
  2. calculate portfolio loss for each trial
  3. use the aggregated trial data to establish a profile of the portfolio's risk characteristics

- ▷ Hypothesis: stock price evolution can be simulated by **geometric Brownian motion** (GBM) with drift
  - constant expected return
  - constant volatility
  - zero transaction costs

- ▷ GBM: a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion
  - stochastic process modeling a "random walk" or "white noise"
  - $W_t - W_s \sim Normal(0, t - s)$

1997 "Nobel-like" prize in economics: Scholes

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Monte Carlo simulation: underlying hypothesis

- Applying the GBM "random walk" model means we are following a weak form of the "efficient market hypothesis"
  - all available public information is already incorporated in the current price
  - the next price movement is conditionally independent of past price movements
- The strong form of the hypothesis says that current price incorporates both public *and private* information

Geometric Brownian motion

drift (instantaneous rate of return on a riskless asset)

time step

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

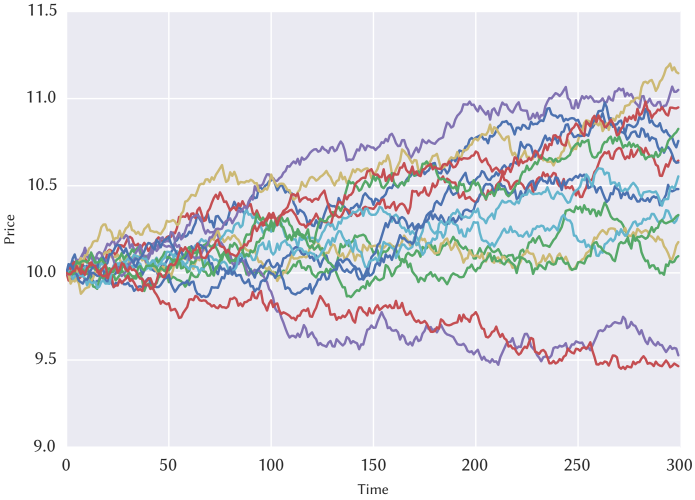
volatility

follows a Normal(0, 1) distribution

where

- S = stock price
- random variable $log(S_t/S_0)$ is normally distributed with mean = $(\mu - \sigma^2/2)t$, variance = $\sigma^2 t$

Monte Carlo simulation: 15 random walks



With large number of simulations, we can estimate:

- ▷ mean final price
- ▷ Value at Risk

→ *slides on Monte Carlo methods at risk-engineering.org*

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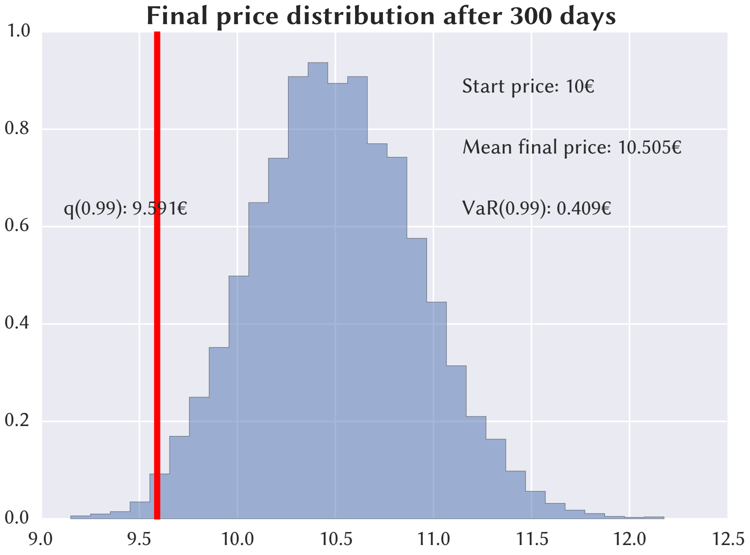
Monte Carlo simulation: histogram of final price



Download the associated Python notebook at `risk-engineering.org`

Note

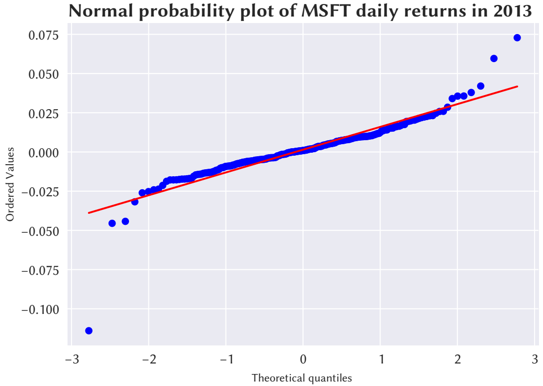
The Black-Scholes model is elegant, but it does not perform very well in practice:

- ▷ it is well known that stock prices jump on occasions and do not always move in the smooth manner predicted by the GBM model
  - Black Tuesday 29 Oct 1929: drop of Dow Jones Industrial Average (DJIA) of 12.8%
  - Black Monday 19 Oct 1987: drop of DJIA of 22.6%
  - Asian and Russian financial crisis of 1997–1998
  - Dot-com bubble burst in 2001
  - Crash of 2008–2009, Covid-19 in 2020
- ▷ stock prices also tend to have fatter tails than those predicted by GBM
- ▷ more sophisticated modelling uses “jump-diffusion” models

> *If the efficient market hypothesis were correct, I’d be a bum in the street with a tin cup.*
>
> *– Warren Buffet*

(Market capitalization of his company Berkshire Hathaway: US$328 billion)

Stock market returns and “fat tails”

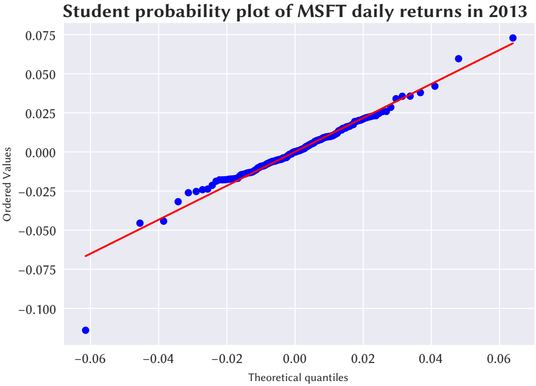


A probability plot shows how the distribution of a sample compares with the distribution of a reference probability distribution by plotting their quantiles against each other.

If distributions are similar, plot will follow a line $Y = X$.

The reference probability distribution is often the normal distribution.

Stock market returns and "fat tails"



Student's t distribution tends to fit stock returns better than a Gaussian (in particular in the tails of the distribution).

The distribution of a random variable X is said to have a "fat tail" if

$$\Pr(X > x) \sim x^{-\alpha} \text{ as } x \to \infty, \quad \alpha > 0$$

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Diversification and portfolios

- Money managers try to reduce their risk exposure by **diversifying** their portfolio of investments
  - attempt to select stocks that have negative correlation: when one goes down, the other goes up
  - same ideas for pooling of risks across business lines and organizations
  - degree of diversification benefit depends on the degree of dependence between pooled risks

> *Diversification benefits can be assessed by correlations between different risk categories. A correlation of +100% means that two variables will fall and rise in lock-step; any correlation below this indicates the potential for diversification benefits.*
>
> *[Treasury and FSA, 2006]*

- Area called "portfolio theory"
  - developed for equities (stocks), but also applied to loans & credits

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Expected returns and risk

- Expected return for an equity i: $\mathbb{E}[R_i] = \mu_i$
  - where μ_i = mean of return distribution for equity i
  - difference between purchase and selling price
- More risk $\longrightarrow$ higher expected return
  - we assume investors are **risk averse**

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Expected returns and risk

Variance

Variance (denoted σ²) is a measure of the **dispersion** of a set of data points **around their mean value**, computed by finding the probability-weighted average of squared deviations from the expected value.

$$\begin{aligned}\sigma_X^2 = Variance(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \mathbb{E}[(X - \mu)^2] \\ &= \sum_{i=1}^{N} p_i (x_i - \mu_X)^2 \quad \text{for a discrete random variable} \\ &= \frac{1}{N}\sum_{i=1}^{N} (x_i - \mu)^2 \quad \text{for a set of } N \text{ equally likely variables}\end{aligned}$$

Variance measures the variability from an average (the *volatility*).

"Risk" in finance is standard deviation of returns for the equity, $\sqrt{variance(i)}$

$$\sigma_i = \sqrt{\mathbb{E}[(\mathbb{E}[R_i] - R_i)^2]}$$

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Expected return and risk: example

- Consider a portfolio of 10 k€ which is invested in equal parts in two instruments:
  - treasury bonds with an annual return of 6%
  - a stock which has a 20% chance of losing half its value and an 80% chance of increasing value by a quarter
- The expected return after one year is the mathematical expectation of the return on the portfolio:
  - expected final value of the bond: $1.06 \times 5000 = 5300$
  - expected final value of the stock: $0.2 \times 2500 + 0.8 \times 6250 = 5500$
  - $\mathbb{E}(\text{return}) = 5400 + 5500 - 10000 = 900$ (= 0.09, or 9%)
- The risk of this investment is the standard deviation of the return

$$\sigma = \sqrt{0.2 \times ((5300 + 2500 - 10000) - 900)^2 + 0.8 \times ((5300 + 6250 - 10000) - 900)^2}$$

$$= 1503.3$$

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Value at Risk of a portfolio

- Remember that $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2cov(X,Y)$
- Variance of a two-stock portfolio:

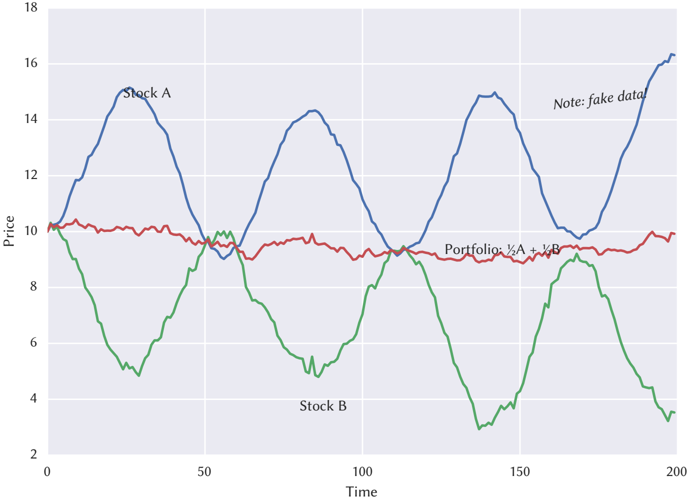
$$\begin{aligned}\sigma^2_{A+B} &= \sigma^2_A + \sigma^2_B + 2\sigma_A\sigma_B\rho_{A,B} \\ &= (\sigma_A + \sigma_B)^2 - 2\sigma_A\sigma_B + 2\rho_{A,B}\sigma_A\sigma_B\end{aligned}$$

  where
  - $\rho_{A,B}$ = covariance (how much do A and B vary together?)
  - σ_i = standard deviation (volatility) of equity i
- Portfolio VaR:

$$\text{VaR}_{A,B} = \sqrt{(\text{VaR}_A + \text{VaR}_B)^2 - 2(1-\rho_{A,B})\text{VaR}_A\text{VaR}_B}$$

Diversification effect: unless the equities are perfectly correlated ($\rho_{A,B} = 1$), the level of risk of a portfolio is smaller than the weighted sum of the two component equities

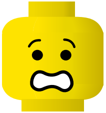
Negatively correlated portfolio reduces risk



Old saying: "Don't put all your eggs in the same basket"

VaR of a three-asset portfolio

- $\text{VaR} = \sqrt{\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + 2\rho_{A,B} + 2\rho_{A,C} + 2\rho_{B,C}}$
- Approach quickly becomes intractable using analytic methods…



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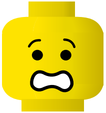
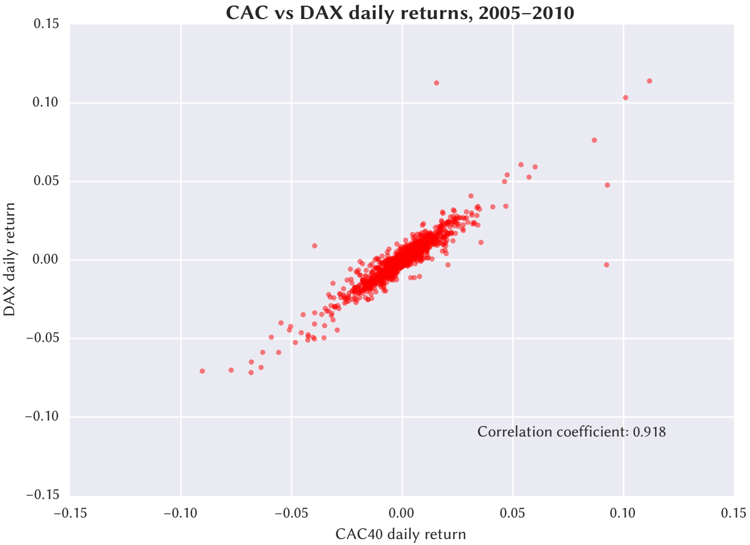


Monte Carlo methods can work, assuming we can generate random returns that are similar to those observed on the market

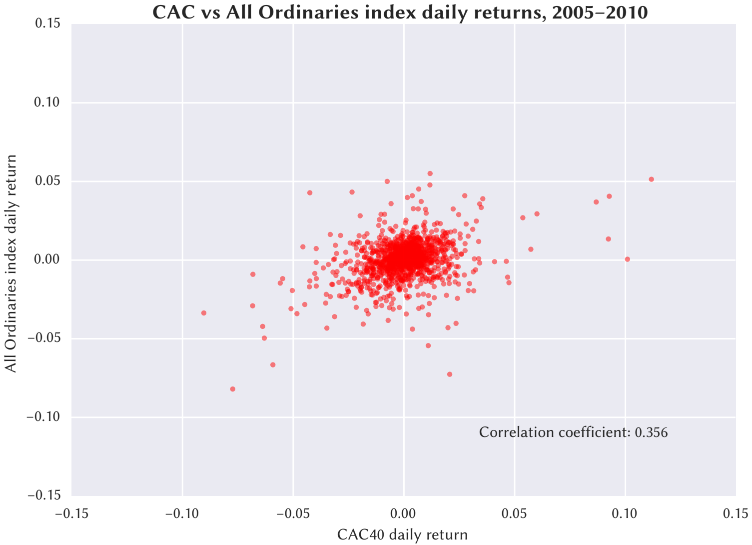
- including the dependencies between stocks…

Example: correlation between stocks



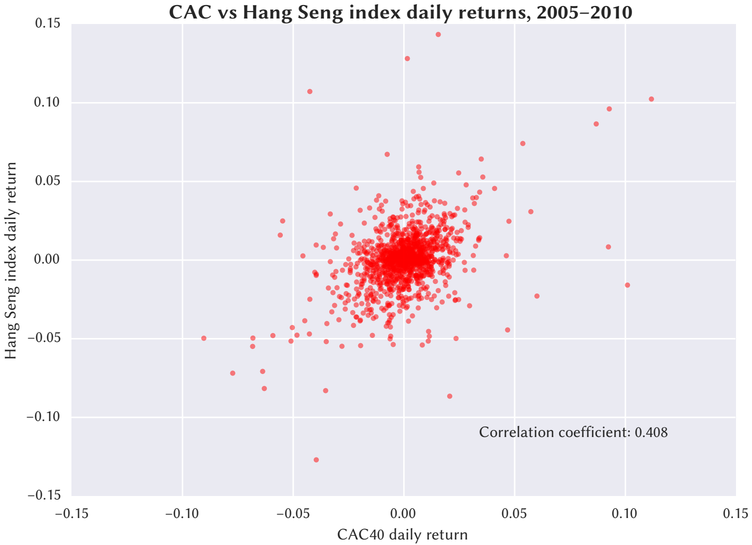
Market opportunities for large French & German firms tend to be strongly correlated, so high correlation between CAC and DAX indices

Example: correlation between stocks



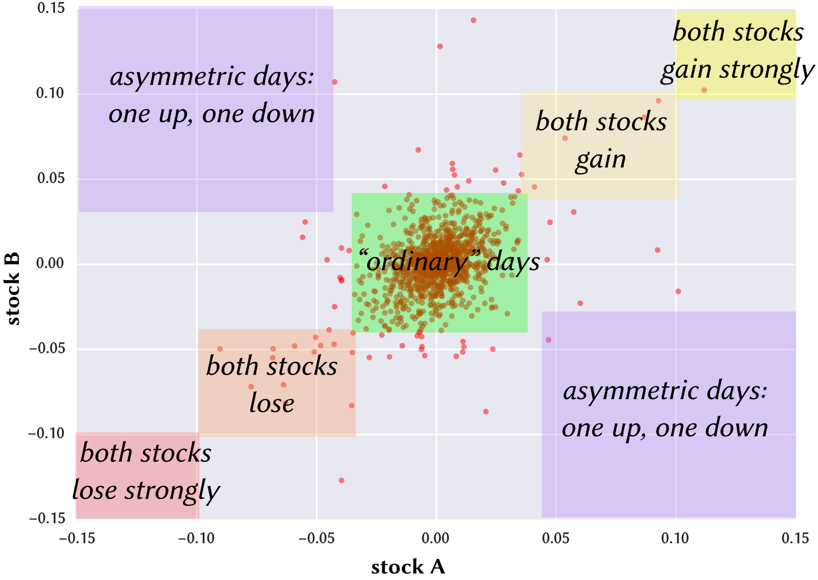
Less market correlation between French & Australian firms, so less index correlation

Example: correlation between stocks



Less market correlation between French & Hong Kong firms, so less index correlation

Correlations and risk: stock portfolios



Simulating correlated random variables

- Let's use the Monte Carlo method to estimate VaR for a portfolio comprising CAC40 and DAX stocks
- We need to generate a large number of simulated daily returns for our CAC40 & DAX portfolio
- We know how to generate daily returns for the CAC40 part of our portfolio
  - simulate random variables from a Student's t distribution with the same mean and standard deviation as the daily returns observed over the last few months for the CAC40
- We can do likewise to generate daily returns for the DAX component
- If our portfolio is equally weighted in CAC40 and DAX, we could try to add together these daily returns to obtain portfolio daily returns

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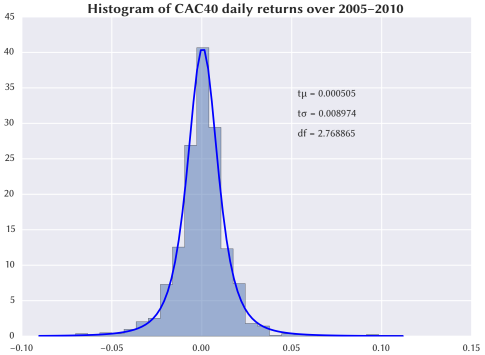
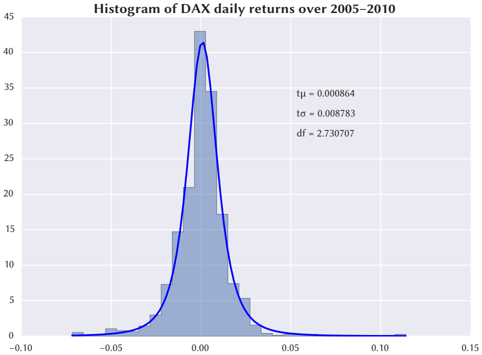
Simulating correlated random variables

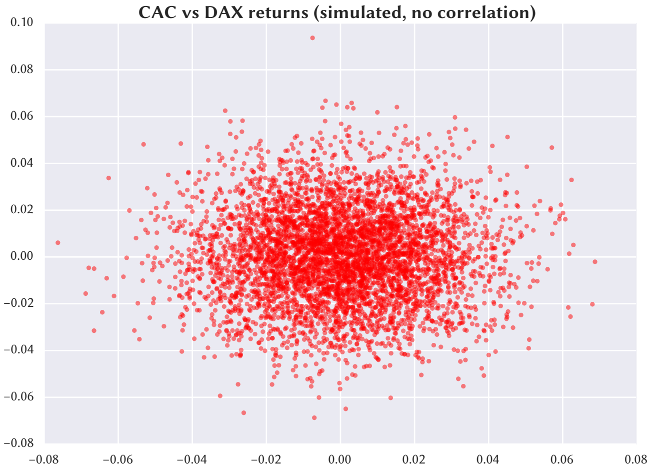


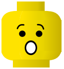


Fit of two Student t distributions to the CAC40 and DAX daily return distribution

Python: `tdf, tmean, tsigma = scipy.stats.t.fit(returns)`

Monte Carlo sampling from these distributions





Problem: our sampling from these random variables doesn't match our observations

We need some way of generating a sample that respects the correlation between the input variables!

Continue with

The mathematical tool we will use to generate samples from correlated random variables is called a *copula*.

To be continued in slides on *Copula and multivariate dependencies* (available on `risk-engineering.org`)

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