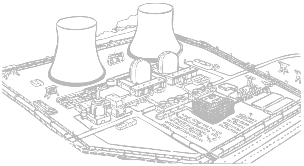


Estimating Value at Risk

Eric Marsden

<eric.marsden@risk-engineering.org>

Do you know how risky your bank is?

Learning objectives

1. Understand measures of financial risk, including *Value at Risk*
2. Understand the impact of correlated risks
3. Know how to use copulas to sample from a multivariate probability distribution, including correlation

⚠ The information presented here is pedagogical in nature and does not constitute investment advice!

Methods used here can also be applied to model natural hazards

RISK ENGINEERING

Warmup. Before reading this material, we suggest you consult the following associated slides:

- ▷ Modelling correlations using Python
- ▷ Statistical modelling with Python

Available from `risk-engineering.org`

Risk in finance

> *There are 10^{11} stars in the galaxy. That used to be a huge number. But it's only a hundred billion. It's less than the national deficit! We used to call them* ***astronomical*** *numbers. Now we should call them* ***economical*** *numbers.*
>
> *— Richard Feynman*

Terminology in finance

Names of some instruments used in finance:

- A **bond** issued by a company or a government is just a loan
  - bond buyer lends money to bond issuer
  - issuer will return money plus some interest when the bond matures
- A **stock** gives you (a small fraction of) ownership in a “listed company”
  - a stock has a price, and can be bought and sold on the stock market
- A **future** is a promise to do a transaction at a later date
  - refers to some “underlying” product which will be bought or sold at a later time
  - example: farmer can sell her crop before harvest, at a fixed price
  - way of transferring risk: farmer protected from risk of price drop, but also from possibility of unexpected profit if price increases

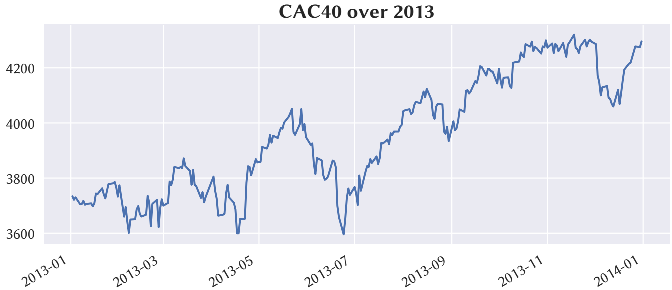
RISK ENGINEERING

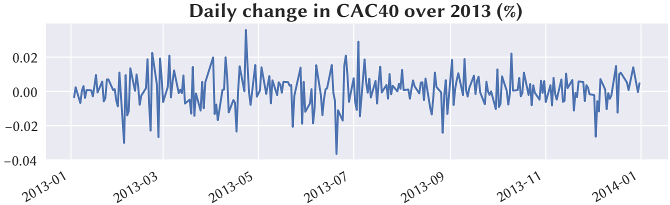
Risk in finance

- Possible definitions:
  - "any event or action that may adversely affect an organization's ability to achieve its objectives and execute its strategies"
  - "the quantifiable likelihood of loss or less-than-expected returns"
- Main categories:
  - **market risk**: change in the value of a financial position due to changes in the value of the underlying components on which that position depends, such as stock and bond prices, exchange rates, commodity prices
  - **credit risk**: not receiving promised repayments on outstanding investments such as loans and bonds, because of the "default" of the borrower
  - **operational risk**: losses resulting from inadequate or failed internal processes, people and systems, or from external events
  - **underwriting risk**: inherent in insurance policies sold, due to changing patterns in natural hazards, in demographic tables (life insurance), in consumer behaviour, and due to systemic risks

Source: *Quantitative Risk Management: Concepts, Techniques and Tools*, A. J. McNeil, R. Frey, P. Embrechts

Stock market returns





Say we have a stock portfolio. How risky is our investment?

We want to model the likelihood that our stock portfolio loses money.

Value at Risk

- Objective: produce a **single number** to summarize my **exposure to market risk**
  - naïve approach: *How much could I lose in the "worst" scenario?*
  - that's a bad question: you could lose everything
- A more informative question:
  - *"What is the loss level that we are X% confident will not be exceeded in N business days?"*
- "5-day $VaR_{0.9}$ = 10 M€" tells us:
  - *I am 90% sure I won't lose more than 10 M€ in the next 5 trading days*
  - *There is 90% chance that my loss will be smaller than 10 M€ in the next 5 days*
  - *There is 10% chance that my loss will be larger than 10 M€ in the next 5 days*
- What it does not tell us:
  - *How much could I lose in those 10% of scenarios?*

RISK ENGINEERING

Value at Risk

Value at risk

A measure of market risk, which uses the statistical analysis of historical market trends and volatilities to estimate the likelihood that a given portfolio's losses (L) will exceed a certain amount l.

$$\mathrm{VaR}_\alpha(L) = \inf\{l \in \mathbb{R} : \Pr(L > l) \leq 1 - \alpha\}$$

where L is the loss of the portfolio and $\alpha \in [0, 1]$ is the confidence level.

Value at Risk

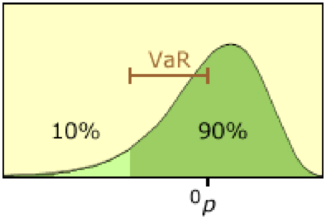
Value at risk

A measure of market risk, which uses the statistical analysis of historical market trends and volatilities to estimate the likelihood that a given portfolio's losses (L) will exceed a certain amount l.

$$\mathrm{VaR}_\alpha(L) = \inf\{l \in \mathbb{R} : \Pr(L > l) \leq 1 - \alpha\}$$

where L is the loss of the portfolio and $\alpha \in [0, 1]$ is the confidence level.

If a portfolio of stocks has a one-day 10% VaR of 1 M€, there is a 10% probability that the portfolio will decline in value by more than 1 M€ over the next day, assuming that markets are normal.



Applications of VaR

- **Risk management**: how much financial risk am I exposed to?
  - Provides a structured methodology for critically thinking about risk, and consolidating risk across an organization
  - VaR can be applied to individual stocks, portfolios of stocks, hedge funds, *etc.*
- **Risk limit setting** (internal controls or regulator imposed)
  - Basel II Accord (financial regulation) attempts to ensure that a bank has adequate capital for the risk that the bank exposes itself to through its lending and investment practices
  - VaR is often used as a measure of market risk
  - Provides a single number which is easy to understand by non-specialists

Limitations of VaR

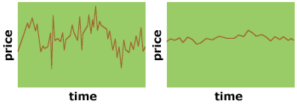
- Typical VaR estimation methods assume "normal" market conditions
- They do not attempt to assess the potential impact of "**black swan**" **events**
  - outlier events that carry an extreme impact
  - example: effects of cascading failure in the banking industry, such as the 2008 subprime mortgage crisis
- More information: see the slides on *Black swans* at `risk-engineering.org`

Alternatives to VaR

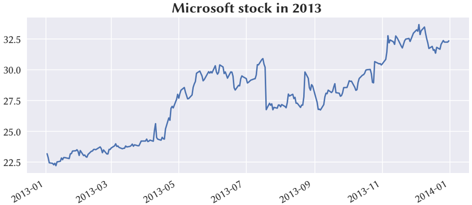
- VaR is a *frequency* measure, not a *severity* measure
  - it's a *threshold*, not an expectation of the amount lost
- Related risk measure: Expected Shortfall, the average loss for losses larger than the VaR
  - expected shortfall at $q\%$ level is the expected return in the worst $q\%$ of cases
  - also called *conditional value at risk* (CVaR) and *expected tail loss*
- Note that
  - ES_q increases as q increases
  - ES_q is always greater than VaR_q at the same q level (for the same portfolio)
- Unlike VaR, expected shortfall is a *coherent* risk measure
  - a risk measure $\mathscr{R}$ is *subadditive* if $\mathscr{R}(X+Y) \leq \mathscr{R}(X) + \mathscr{R}(Y)$
  - the risk of two portfolios combined cannot exceed the risk of the two separate portfolios added together (diversification does not increase risk)

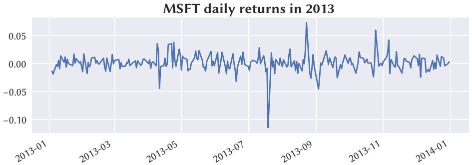
Estimating VaR

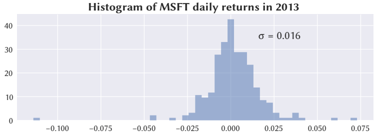
- Estimation is difficult because we are dealing with **rare events** whose probability distribution is unknown
- Three main methods are used to estimate VaR:
  1. historical bootstrap method
  2. variance-covariance method
  3. Monte Carlo simulation
- All are based on estimating **volatility**
- Applications of the *constant expected return* model which is widely used in finance
  - assumption: an asset's return over time is independent and identically normally distributed with a constant (time invariant) mean and variance



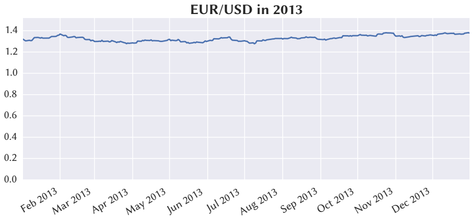
Understanding volatility

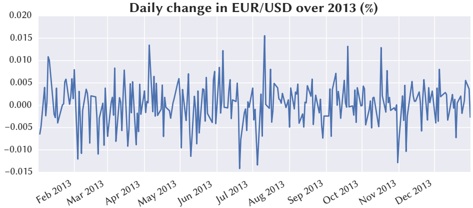


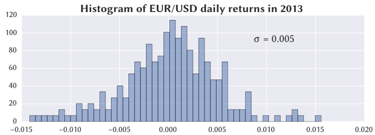




high volatility



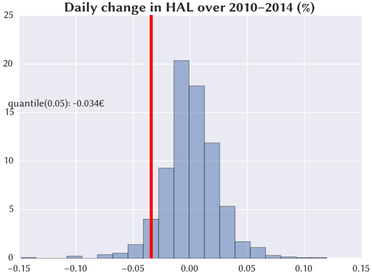




low volatility

Historical bootstrap method

- Hypothesis: history is representative of future activity
- Method: calculate **empirical quantiles** from a histogram of daily returns
- 0.05 empirical quantile of daily returns is at -0.034:
  - with 95% confidence, our worst daily loss will not exceed 3.4%
  - 1 M€ investment: one-day 5% VaR is 0.034 × 1 M€ = 34 k€
  - (note: the 0.05 quantile is the 5$^{\text{th}}$ percentile)
- 0.01 empirical quantile of daily returns is at -0.062:
  - with 99% confidence, our worst daily loss will not exceed 6.2%
  - 1 M€ investment: one-day 1% VaR is 0.062 × 1 M€ = 62 k€

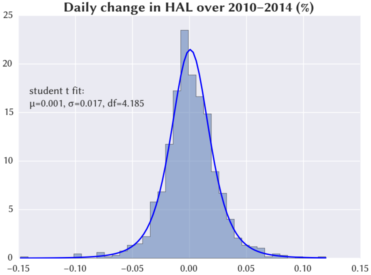


Download this content as a Python notebook at `risk-engineering.org`

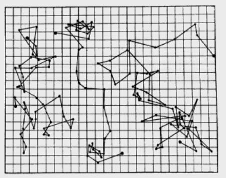
RISK ENGINEERING

Variance-covariance method

- ▷ Hypothesis: daily returns are normally distributed
- ▷ Method: analytic quantiles by **curve fitting to historical data**
  - here: Student's t distribution
- ▷ 0.05 analytic quantile is at -0.0384
  - with 95% confidence, our worst daily loss will not exceed 3.84%
  - 1 M€ investment: one-day 5% VaR is 0.0384×1 M€ = 38 k€
- ▷ 0.01 analytic quantile is at -0.0546
  - with 99% confidence, our worst daily loss will not exceed 5.46%
  - 1 M€ investment: one-day 1% VaR is 0.0546×1 M€ = 54 k€



Monte Carlo simulation



- ▷ Method:
  1. run many “trials” with random market conditions
  2. calculate portfolio loss for each trial
  3. use the aggregated trial data to establish a profile of the portfolio’s risk characteristics

- ▷ Hypothesis: stock price evolution can be simulated by **geometric Brownian motion** (GBM) with drift
  - constant expected return
  - constant volatility
  - zero transaction costs

- ▷ GBM: a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion
  - stochastic process modeling a “random walk” or “white noise”
  - $W_t - W_s \sim Normal(0, t - s)$

1997 “Nobel-like” prize in economics: Scholes

Monte Carlo simulation: underlying hypothesis

- Applying the GBM "random walk" model means we are following a weak form of the "efficient market hypothesis"
  - all available public information is already incorporated in the current price
  - the next price movement is conditionally independent of past price movements
- The strong form of the hypothesis says that current price incorporates both public *and private* information

Geometric Brownian motion

drift (instantaneous rate of return on a riskless asset)

time step

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

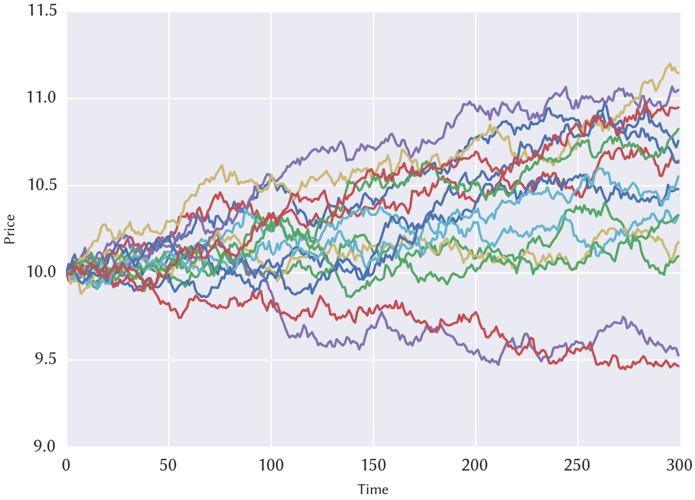
volatility

follows a Normal(∅, 1) distribution

where

- S = stock price
- random variable $log(S_t/S_0)$ is normally distributed with mean = $(\mu - \sigma^2/2)t$, variance = $\sigma^2 t$

Monte Carlo simulation: 15 random walks

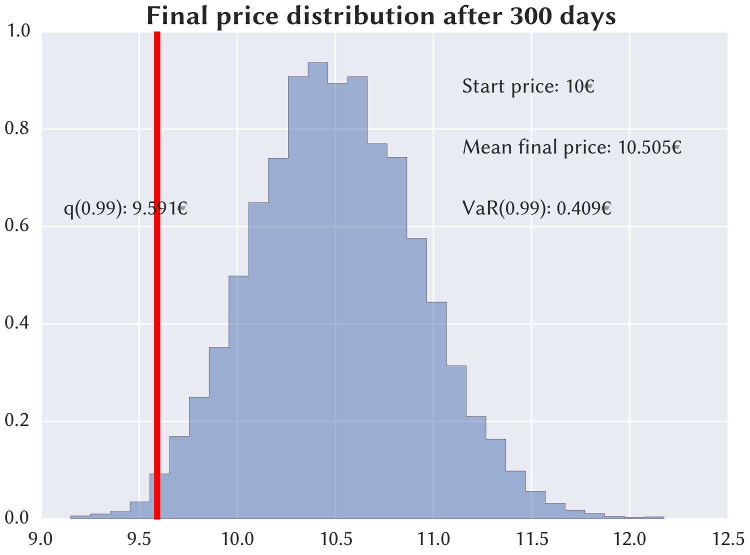


With large number of simulations, we can estimate:

- ▷ mean final price
- ▷ Value at Risk

→ *slides on Monte Carlo methods at risk-engineering.org*

Monte Carlo simulation: histogram of final price



Download the associated Python notebook at `risk-engineering.org`

Note

The Black-Scholes model is elegant, but it does not perform very well in practice:

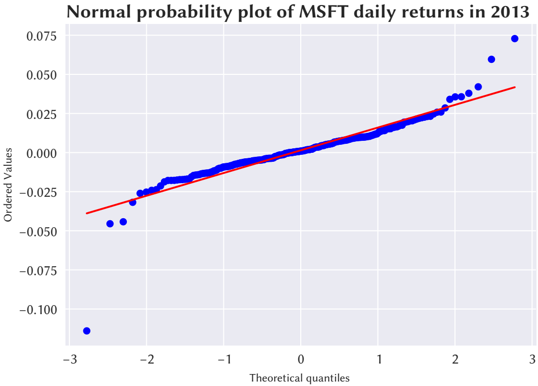
- ▷ it is well known that stock prices jump on occasions and do not always move in the smooth manner predicted by the GBM model
  - Black Tuesday 29 Oct 1929: drop of Dow Jones Industrial Average (DJIA) of 12.8%
  - Black Monday 19 Oct 1987: drop of DJIA of 22.6%
  - Asian and Russian financial crisis of 1997–1998
  - Dot-com bubble burst in 2001
  - Crash of 2008–2009, Covid-19 in 2020
- ▷ stock prices also tend to have fatter tails than those predicted by GBM
- ▷ more sophisticated modelling uses "jump-diffusion" models

> *If the efficient market hypothesis were correct, I'd be a bum in the street with a tin cup.*
>
> – *Warren Buffet*

(Market capitalization of his company Berkshire Hathaway: US$328 billion)

RISK ENGINEERING

Stock market returns and "fat tails"

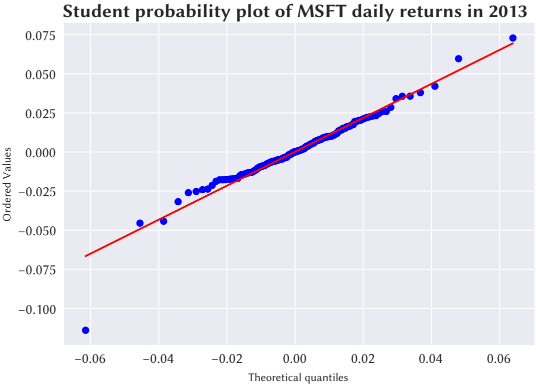


A probability plot shows how the distribution of a sample compares with the distribution of a reference probability distribution by plotting their quantiles against each other.

If distributions are similar, plot will follow a line $Y = X$.

The reference probability distribution is often the normal distribution.

Stock market returns and "fat tails"



Student's t distribution tends to fit stock returns better than a Gaussian (in particular in the tails of the distribution).

The distribution of a random variable X is said to have a "fat tail" if

$$\Pr(X > x) \sim x^{-\alpha} \text{ as } x \to \infty, \quad \alpha > 0$$

Diversification and portfolios

- Money managers try to reduce their risk exposure by **diversifying** their portfolio of investments
  - attempt to select stocks that have negative correlation: when one goes down, the other goes up
  - same ideas for pooling of risks across business lines and organizations
  - degree of diversification benefit depends on the degree of dependence between pooled risks

> *Diversification benefits can be assessed by correlations between different risk categories. A correlation of +100% means that two variables will fall and rise in lock-step; any correlation below this indicates the potential for diversification benefits.*
>
> *[Treasury and FSA, 2006]*

- Area called "portfolio theory"
  - developed for equities (stocks), but also applied to loans & credits

RISK ENGINEERING

Expected returns and risk

- Expected return for an equity i: $\mathbb{E}[R_i] = \mu_i$
  - where μ_i = mean of return distribution for equity i
  - difference between purchase and selling price
- More risk $\longrightarrow$ higher expected return
  - we assume investors are **risk averse**

RISK ENGINEERING

Expected returns and risk

Variance

Variance (denoted σ^2) is a measure of the **dispersion** of a set of data points **around their mean value**, computed by finding the probability-weighted average of squared deviations from the expected value.

$$\begin{aligned}
\sigma_X^2 = Variance(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[(X - \mu)^2] \\
&= \sum_{i=1}^{N} p_i (x_i - \mu_X)^2 \quad \text{for a discrete random variable} \\
&= \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2 \quad \text{for a set of } N \text{ equally likely variables}
\end{aligned}$$

Variance measures the variability from an average (the *volatility*).

"Risk" in finance is standard deviation of returns for the equity, $\sqrt{variance(i)}$

$$\sigma_i = \sqrt{\mathbb{E}[(\mathbb{E}[R_i] - R_i)^2]}$$

Expected return and risk: example

- Consider a portfolio of 10 k€ which is invested in equal parts in two instruments:
  - treasury bonds with an annual return of 6%
  - a stock which has a 20% chance of losing half its value and an 80% chance of increasing value by a quarter
- The expected return after one year is the mathematical expectation of the return on the portfolio:
  - expected final value of the bond: 1.06 × 5000 = 5300
  - expected final value of the stock: 0.2 × 2500 + 0.8 × 6250 = 5500
  - $\mathbb{E}$(return) = 5400 + 5500 - 10000 = 900 (= 0.09, or 9%)
- The risk of this investment is the standard deviation of the return

$$\sigma = \sqrt{0.2 \times ((5300 + 2500 - 10000) - 900)^2 + 0.8 \times ((5300 + 6250 - 10000) - 900)^2}$$

$$= 1503.3$$

RISK ENGINEERING

Value at Risk of a portfolio

- Remember that $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2cov(X,Y)$
- Variance of a two-stock portfolio:

$$\begin{aligned}\sigma^2_{A+B} &= \sigma^2_A + \sigma^2_B + 2\sigma_A\sigma_B\rho_{A,B} \\ &= (\sigma_A + \sigma_B)^2 - 2\sigma_A\sigma_B + 2\rho_{A,B}\sigma_A\sigma_B\end{aligned}$$

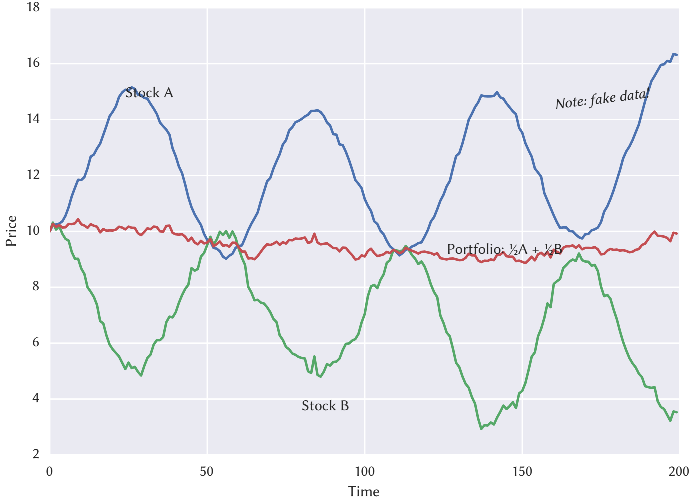
  where
  - $\rho_{A,B}$ = covariance (how much do A and B vary together?)
  - σ_i = standard deviation (volatility) of equity i
- Portfolio VaR:

$$\text{VaR}_{A,B} = \sqrt{(\text{VaR}_A + \text{VaR}_B)^2 - 2(1-\rho_{A,B})\text{VaR}_A\text{VaR}_B}$$

Diversification effect: unless the equities are perfectly correlated ($\rho_{A,B} = 1$), the level of risk of a portfolio is smaller than the weighted sum of the two component equities

RISK ENGINEERING

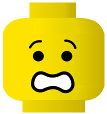
Negatively correlated portfolio reduces risk



Old saying: "Don't put all your eggs in the same basket"

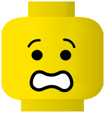
VaR of a three-asset portfolio

- $\text{VaR} = \sqrt{\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + 2\rho_{A,B} + 2\rho_{A,C} + 2\rho_{B,C}}$
- Approach quickly becomes intractable using analytic methods…



VaR of a three-asset portfolio

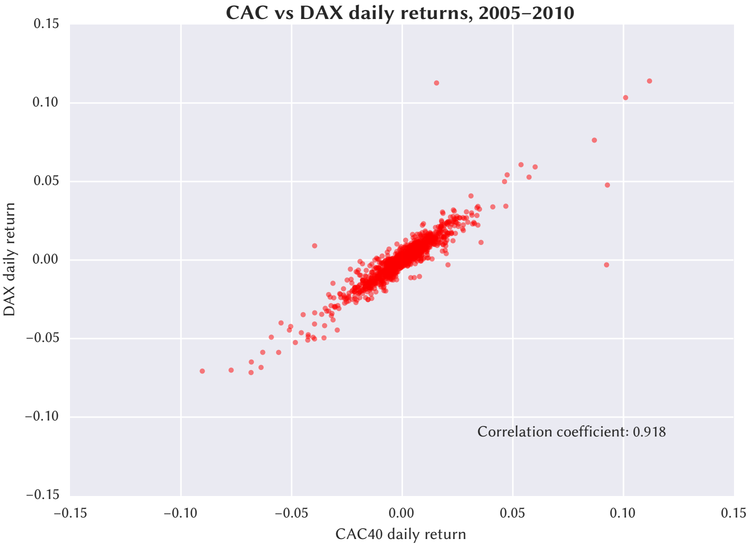
- $\text{VaR} = \sqrt{\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + 2\rho_{A,B} + 2\rho_{A,C} + 2\rho_{B,C}}$
- Approach quickly becomes intractable using analytic methods…



Monte Carlo methods can work, assuming we can generate random returns that are similar to those observed on the market

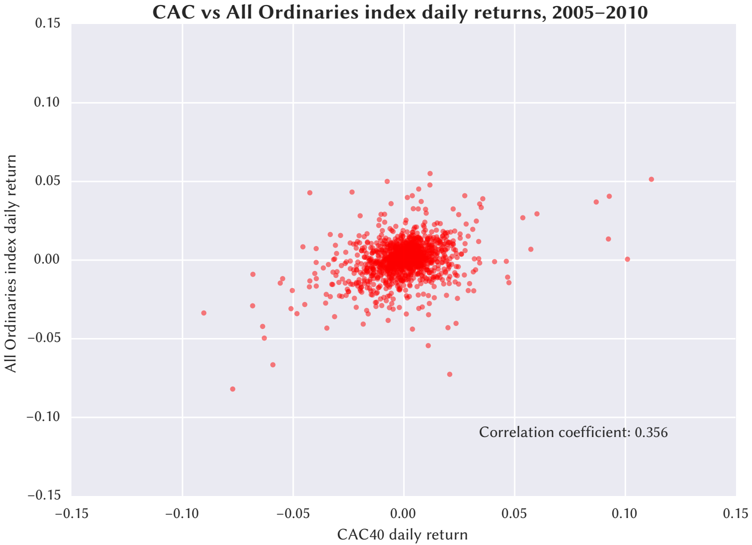
- including the dependencies between stocks…

Example: correlation between stocks



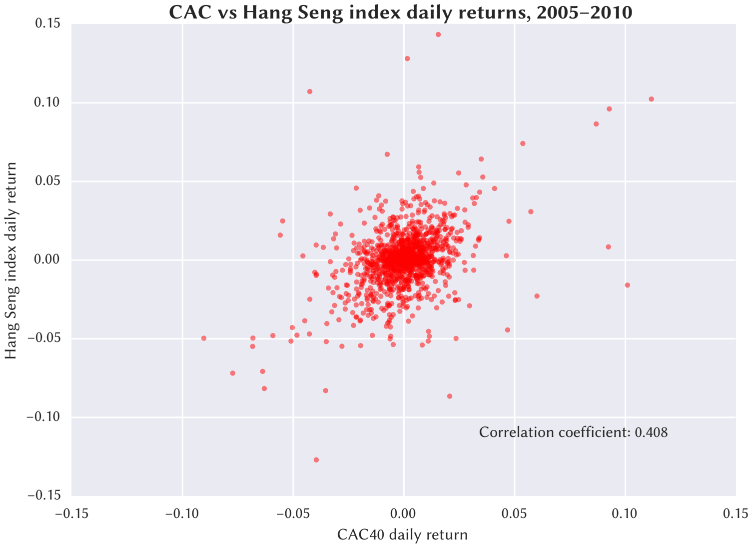
Market opportunities for large French & German firms tend to be strongly correlated, so high correlation between CAC and DAX indices

Example: correlation between stocks



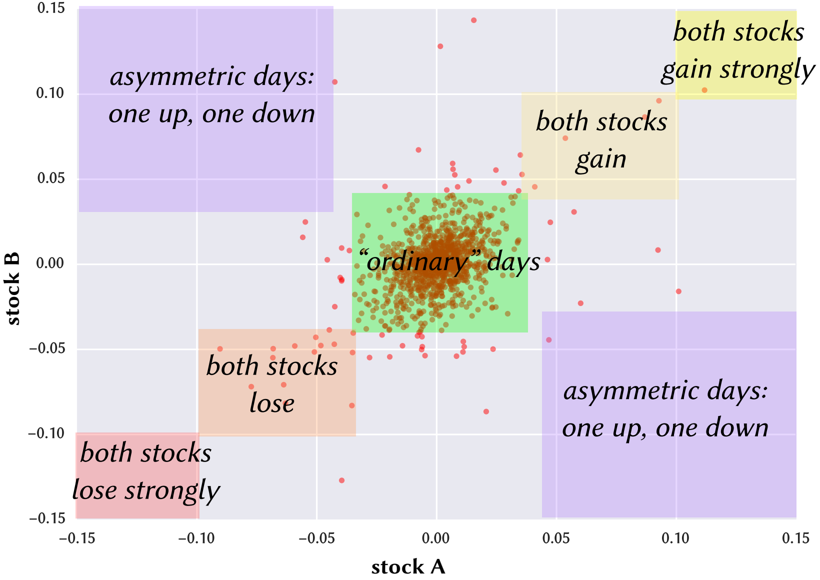
Less market correlation between French & Australian firms, so less index correlation

Example: correlation between stocks



Less market correlation between French & Hong Kong firms, so less index correlation

Correlations and risk: stock portfolios



Simulating correlated random variables

- Let's use the Monte Carlo method to estimate VaR for a portfolio comprising CAC40 and DAX stocks

- We need to generate a large number of simulated daily returns for our CAC40 & DAX portfolio

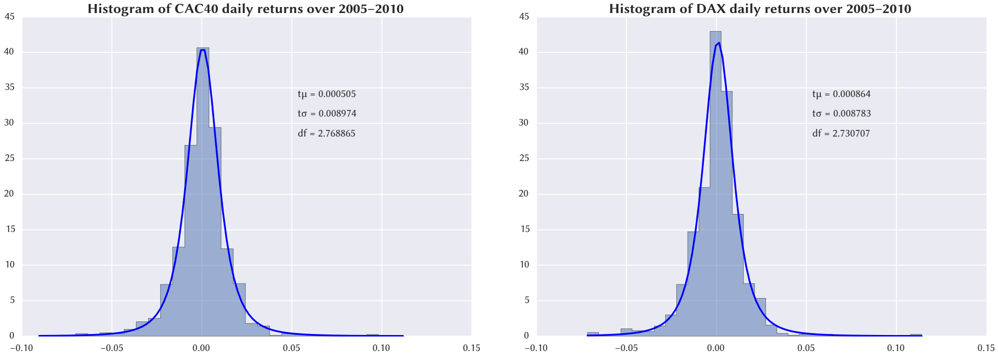
- We know how to generate daily returns for the CAC40 part of our portfolio
  - simulate random variables from a Student's t distribution with the same mean and standard deviation as the daily returns observed over the last few months for the CAC40

- We can do likewise to generate daily returns for the DAX component

- If our portfolio is equally weighted in CAC40 and DAX, we could try to add together these daily returns to obtain portfolio daily returns

RISK ENGINEERING

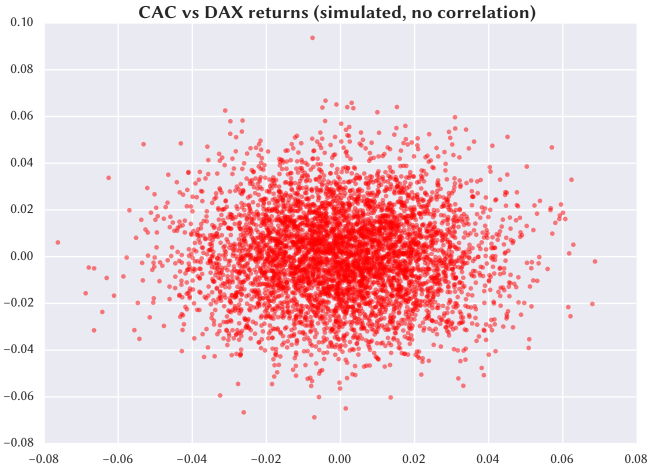
Simulating correlated random variables

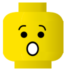


Fit of two Student t distributions to the CAC40 and DAX daily return distribution

Python: `tdf, tmean, tsigma = scipy.stats.t.fit(returns)`

Monte Carlo sampling from these distributions





Problem: our sampling from these random variables doesn't match our observations

We need some way of generating a sample that respects the correlation between the input variables!

Continue with

The mathematical tool we will use to generate samples from correlated random variables is called a *copula*.

To be continued in slides on *Copula and multivariate dependencies* (available on `risk-engineering.org`)

For more free content on risk engineering, visit `risk-engineering.org`

Image credits

- Cat stretching (slide 3): norsez via `flic.kr/p/e8q1GE`, CC BY-NC-ND licence
- Brownian motion (slide 16), reproduced from Jean Baptiste Perrin, "Mouvement brownien et réalité moléculaire", Ann. de Chimie et de Physique (VIII) 18, 5-114, 1909 (public domain)

For more free content on risk engineering, visit `risk-engineering.org`

Feedback welcome!

Was some of the content unclear? Which parts were most useful to you? Your comments to `feedback@risk-engineering.org` (email) will help us to improve these materials. Thanks!

For more free content on risk engineering, visit `risk-engineering.org`