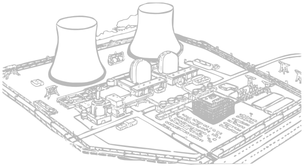

Monte Carlo methods for risk analysis

Eric Marsden

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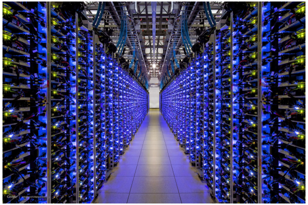
Context



- Some problems cannot be expressed in analytical form
- Some problems are difficult to define in a deterministic manner
- Modern computers are amazingly fast
- Allow you to run "numerical experiments" to see what happens "on average" over a large number of runs
  - also called *stochastic simulation*

Etymological note: stochastic is a synonym for probabilistic: the former comes from the Greek word stokházomai, "aiming at a target, guessing" and the latter from the Latin term probābilis ("probable, credible")

Monte Carlo simulation

- ▷ Monte Carlo method: computational method using repeated random sampling to obtain numerical results
  - named after gambling in casinos
- ▷ Technique invented during the Manhattan project (US nuclear bomb development)
  - their development coincides with invention of electronic computers, which greatly accelerated repetitive numerical computations
- ▷ Widely used in engineering, finance, business, project planning
- ▷ Implementation with computers uses **pseudo-random number generators**

  → `random.org/randomness/`

Photo: Monte Carlo casino

Monte Carlo simulation: steps

Define **possible inputs**

Generate inputs randomly

Computation on the inputs

Aggregate the results

Define the **domain of possible inputs**.

The simulated "universe" should be similar to the universe whose behavior we wish to describe and investigate.

Monte Carlo simulation: steps

Define **possible inputs**

Generate inputs randomly

Computation on the inputs

Aggregate the results

Generate inputs randomly from a probability distribution over the domain

- inputs should be generated so that their characteristics are similar to the real universe we are trying to simulate
- in particular, dependencies between the inputs should be represented

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Monte Carlo simulation: steps

Define **possible inputs**

Generate inputs randomly

Computation on the inputs

Aggregate the results

The computation should be **deterministic**.

Monte Carlo simulation: steps

Define **possible inputs**

Generate inputs randomly

Computation on the inputs

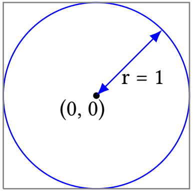
Aggregate the results

Aggregate the results to obtain the output of interest.

Typical outputs:

- histogram
- summary statistics (mean, standard deviation...)
- confidence intervals

Example: estimate the value of pi



- Consider the largest circle which can be fit in the square ranging on $\mathbb{R}^2$ over $[-1, 1]^2$
  - the circle has radius 1 and area π
  - the square has an area of $2^2 = 4$
  - the ratio between their areas is thus $\frac{\pi}{4}$
- We can approximate the value of π using the following Monte Carlo procedure:
  1. draw the square over $[-1, 1]^2$
  2. draw the largest circle that fits inside the square
  3. randomly scatter a large number N of grains of rice over the square
  4. count how many grains fell inside the circle
  5. the count divided by N and multiplied by 4 is an approximation of π

Example: estimate the value of pi

Define **possible inputs**

Generate inputs randomly

Computation on the inputs

Aggregate the results

All points within the $[-1, 1]^2$ unit square, uniformly distributed.

Example: estimate the value of pi

Define **possible inputs**

Generate inputs randomly

Computation on the inputs

Aggregate the results

Generate one point `(x, y)` from the unit square in Python:

```
x = numpy.random.uniform(-1, 1)
y = numpy.random.uniform(-1, 1)
```

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Example: estimate the value of pi

Define **possible inputs**

Generate inputs randomly

Computation on the inputs

Aggregate the results

Test whether a randomly generated point `(x, y)` is within the circle:

```
if numpy.sqrt(x**2 + y**2) < 1:
    print("The point is inside")
```

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Example: estimate the value of pi

Define **possible inputs**

Generate inputs randomly

Computation on the inputs

Aggregate the results

Count the proportion of points that are within the circle:

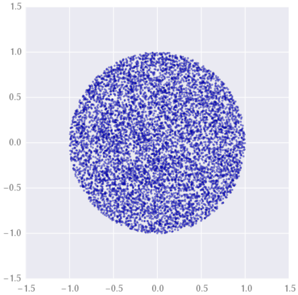
```
N = 10_000
inside = 0
for i in range(N):
    x = numpy.random.uniform(-1, 1)
    y = numpy.random.uniform(-1, 1)
    if numpy.sqrt(x**2 + y**2) < 1:
        inside += 1
p = inside / float(N)
print("Proportion inside: {}".format(p))
```

Example: estimate the value of pi

Putting it all together: here is an implementation in Python with the NumPy library.

```
import numpy

N = 10_000
inside = 0
for i in range(N):
    x = numpy.random.uniform(-1, 1)
    y = numpy.random.uniform(-1, 1)
    if numpy.sqrt(x**2 + y**2) < 1:
        inside += 1
print(4*inside/float(N))
3.142
```



Download the associated Python notebook at `risk-engineering.org`

Exercise: speed of convergence

- Mathematical theory states that the **error** of a Monte Carlo estimation technique should decrease proportionally to the **square root of the number of trials**
- **Exercise**: modify the Monte Carlo procedure for estimation of π
  - within the loop, calculate the current estimation of π
  - calculate the error of this estimation
  - plot the error against the square root of the current number of iterations

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Why does it work?

- The **law of large numbers** describes what happens when performing the same experiment many times
- After many trials, the average of the results should be close to the expected value
  - increasing the number of trials will increase accuracy
- For Monte Carlo simulation, this means that we can learn properties of a random variable (mean, variance, *etc.*) simply by simulating it over many trials

Example: coin flipping

Flip a coin 10 times. What is the probability of getting more than 3 heads?

Example: coin flipping

Flip a coin 10 times. What is the probability of getting more than 3 heads?

Analytical solution

Let's try to remember how the binomial distribution works. Here we have $n = 10$, $p = 0.5$ and `cdf(3)` is the probability of seeing three or fewer heads.

```
> from scipy.stats import binom
> throws = binom(n=10, p=0.5)
> 1 - throws.cdf(3)
0.828125
```

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Example: coin flipping

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```

Monte Carlo simulation

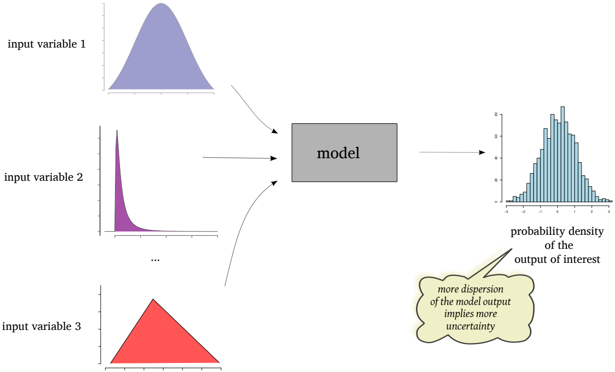
Just simulate the coin flip sequence a million times and count the simulations where we have more than 3 heads.

```
import numpy
def headcount():
    tosses = numpy.random.uniform(0, 1, 10)
    return (tosses > 0.5).sum()

N = 1_000_000
count = 0
for i in range(N):
    if headcount() > 3: count += 1
count / float(N)
0.828117
```

Application to uncertainty analysis

- Uncertainty analysis: propagate uncertainty on input variables through the model to obtain a probability distribution of the output(s) of interest
- Uncertainty on each input variable is characterized by a probability density function
- Run the model a large number of times with input values drawn randomly from their PDF
- Aggregate the output uncertainty as a probability distribution



→ *slides on uncertainty in risk analysis at risk-engineering.org*

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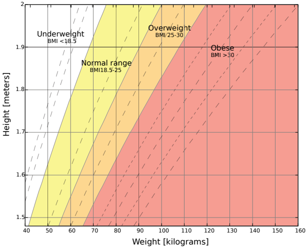
A simple application in uncertainty propagation



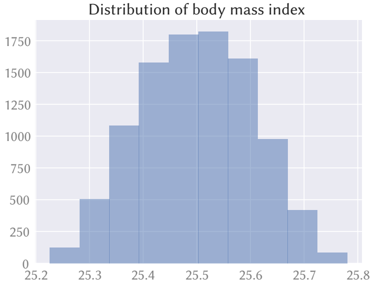
- The **body mass index** (BMI) is the ratio $\frac{\text{body mass } (kg)}{\text{body height } (m)^2}$
  - often used as an (imperfect) indicator of obesity or malnutrition
- **Task**: calculate your BMI and the associated uncertainty interval, assuming:
  - your weight scale tells you that you weigh 84 kg (precision shown to the nearest kilogram)
  - a tape measure says you are between 181 and 182 cm tall (most likely value is 181.5 cm)
- We will run a Monte Carlo simulation on the model BMI $= \frac{m}{h^2}$ with
  - m drawn from a $U(83.5, 84.5)$ uniform distribution
  - h drawn from a $T(1.81, 1.815, 1.82)$ triangular distribution

A simple application in uncertainty propagation

```
import numpy
from numpy.random import *
import matplotlib.pyplot as plt

N = 10_000
def BMI():
    m = uniform(83.5, 84.5)
    h = triangular(1.81, 1.815, 1.82)
    return m / h**2

sim = numpy.zeros(N)
for i in range(N):
    sim[i] = BMI()
plt.hist(sim)
```



A simple application in uncertainty propagation

- Note: analytical estimation of the output uncertainty would be difficult even on this trivial example
- With more than two input probability distributions, becomes *very* difficult
- Quantile measures, often needed for risk analysis, are often difficult to calculate analytically
  - *"what is the 95th percentile of the high water level?"*
- Monte Carlo techniques are a **simple and convenient** way to obtain these numbers
  - express the problem in a direct way and **let the computer do the hard work**!

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Second example for uncertainty propagation

- X and Y are both uniformly distributed over $[0, 100]$
- We are interested in the distribution of $Z = X \times Y$
- **Q**: What is the 95$^{\text{th}}$ percentile of Z?

```
N = 10_000
zs = numpy.zeros(N)
for i in range(N):
    x = numpy.random.uniform(0, 100)
    y = numpy.random.uniform(0, 100)
    zs[i] = x * y
numpy.percentile(zs, 95)
```

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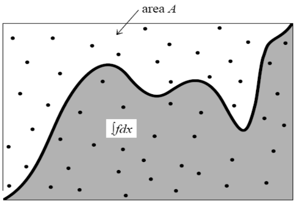
Application in resolving numerical integrals



- Assume we want to evaluate an integral $\int_I f(x)\mathrm{d}x$
- **Principle**: the integral to compute is related to the expectation of a random variable

$$\mathbb{E}(f(X)) = \int_I f(x)\mathrm{d}x$$

- **Method**:
  - Sample points within I
  - Calculate the mean of the random variable within I
  - Integral = sampled area × mean
- **Advantages**: the method works even without knowing the analytical form of f, and also if f is not continuous

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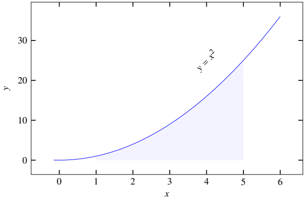
Trivial integration example



Task: find the shaded area, $\int_1^5 x^2 \, \mathrm{d}x$

Analytical solution

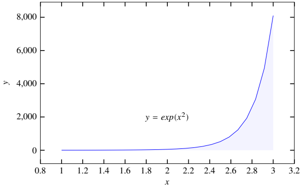
```
import sympy
x = sympy.Symbol("x")
i = sympy.integrate(x**2)
i.subs(x, 5) - i.subs(x, 1)
124/3
float(i.subs(x, 5) - i.subs(x, 1))
41.333333333333336
```

Numerical solution

```
N = 100_000
accum = 0
for i in range(N):
    x = numpy.random.uniform(1, 5)
    accum += x**2
area = 4
integral = area * accum / float(N)
41.278
```

The SymPy library for symbolic mathematics in Python is available from sympy.org

Simple integration example



Task: find the shaded area, $\int_1^3 e^{x^2}\,\mathrm{d}x$

Analytical solution

```
import sympy
x = sympy.Symbol("x")
i = sympy.integrate(sumpy.exp(x**2))
i.subs(x, 3) - i.subs(x, 1)
-sqrt(pi)*erfi(1)/2 + sqrt(pi)*erfi(3)/2
float(i.subs(x, 3) - i.subs(x, 1))
1443.082471146807
```

Numerical solution

```
N = 100_000
accum = 0
for i in range(N):
    x = numpy.random.uniform(1, 3)
    accum += numpy.exp(x**2):
        count += 1
area = 3 - 1
integral = area * accum / float(N)
1451.3281492713274
```

The SymPy library for symbolic mathematics in Python is available from sympy.org

2D integration example

Task: resolve the double integral

$$\int_0^1 \int_4^6 cos(x^4) + 3y^2 \, dx \, dy$$

Analytical solution

```
import sympy
x = sympy.Symbol("x")
y = sympy.Symbol("y")
d1 = sympy.integrate(sympy.cos(x**4) + 3 * y**2, x)
d2 = sympy.integrate(d1.subs(x, 6) - d1.subs(x, 4), y)
sol = d2.subs(y, 1) - d2.subs(y, 0)
float(sol)
2.005055086749674
```

Numerical solution

```
N = 100_000
accum = 0
for i in range(N):
    x = numpy.random.uniform(4, 6)
    y = numpy.random.uniform(0, 1)
    accum += numpy.cos(x**4) + 3 * y * y
volume = 2 * 1
integral = volume * accum/float(N)
2.0100840446967103
```

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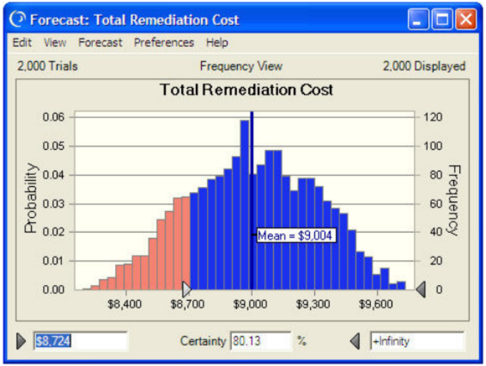
Relevant tools

(if you can't use Python...)

Relevant commercial tools

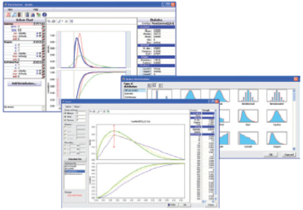


Example tools with Excel integration:

- Palisade TopRank®
- Oracle Crystal Ball®

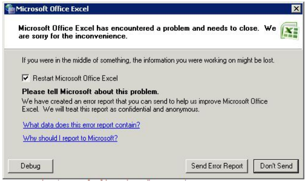
Typically quite expensive…

Free plugins for Microsoft Excel



- ▷ A free Microsoft Excel plugin from Vose Software
  - `vosesoftware.com/products/modelrisk/`
  - "standard" version is free (requires registration)
- ▷ Simtools, a free add-in for Microsoft Excel by R. Myerson, professor at the University of Chicago
  - `home.uchicago.edu/ rmyerson/addins.htm`
- ▷ MonteCarlito, a free add-in for Microsoft Excel
  - `montecarlito.com`
  - distributed under the terms of the GNU General Public Licence

Beware the risks of Excel!



- ▷ Student finds serious errors in austerity research undertaken by Reinhart and Rogoff (cells left out of calculations of averages…)
- ▷ JP Morgan underestimates value at risk due to a spreadsheet error
- ▷ London 2012 Olympics: organization committee oversells synchronized swimming events by 10 000 tickets
- ▷ Cement factory receives 350 000 USD fine for a spreadsheet error (2011, Arizona)

Source: European Spreadsheet Risks Interest Group, `eusprig.org/horror-stories.htm`

Sampling methods

Latin Hypercube Sampling

- ▷ With standard random sampling, you may end up with samples unevenly spread out over the input space
- ▷ Latin Hypercube Sampling (LHS):
  - split up each input variable into a number of equiprobable intervals
  - sample separately from each interval
- ▷ Also called *stratified sampling without replacement*
- ▷ Typically leads to faster convergence than Monte Carlo procedures using standard random sampling

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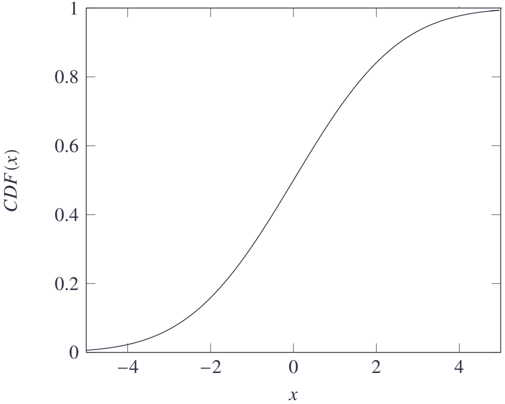
Sampling methods illustrated

Standard sampling (one dimensional):

1. generate a random number from a uniform distribution between 0 and 1
2. use the inverse CDF of the target distribution (the percentile function) to calculate the corresponding output
3. repeat

Illustrated to the right with the normal distribution.



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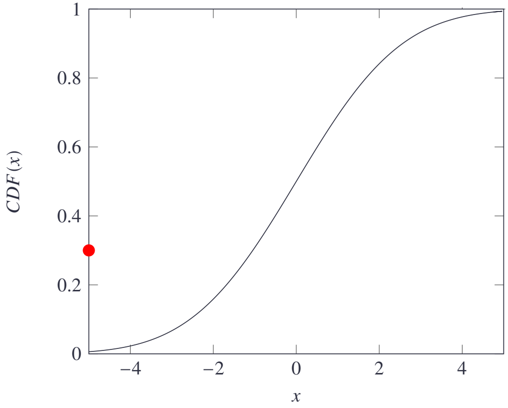

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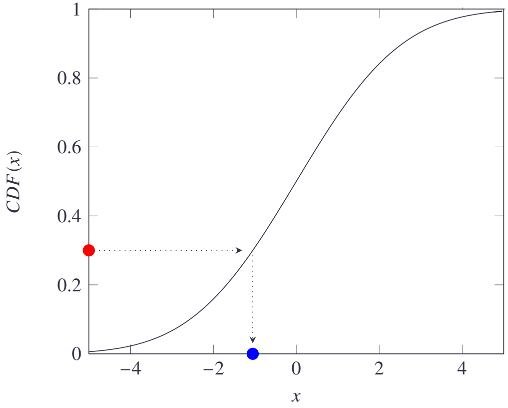

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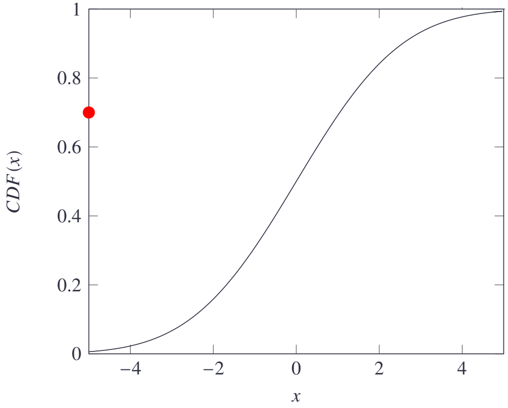

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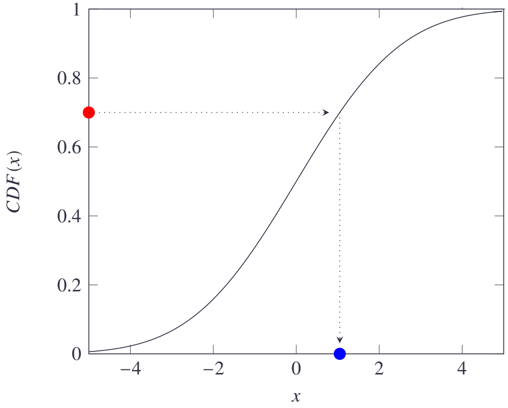
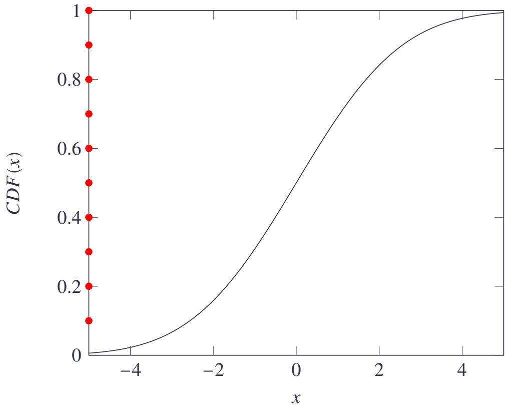


Latin Hypercube Sampling: illustration

Latin hypercube sampling (one dimensional):

1. split the [0,1] interval into 10 equiprobable intervals
2. propagate via the inverse CDF to the output distribution
3. take $N/10$ standard samples from each interval of the output distribution

Illustrated to the right with the normal distribution.



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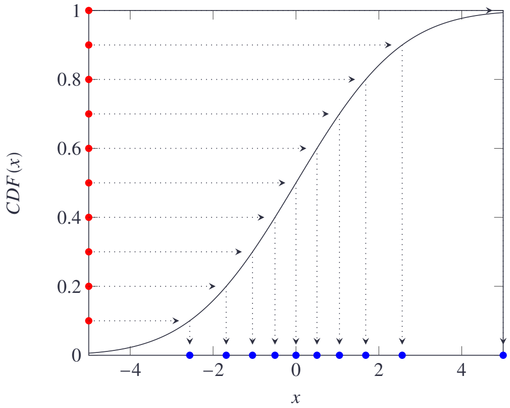

Note: spacing of red points is regular; more space between blue points near the tails of the distribution (where probability density is lower)

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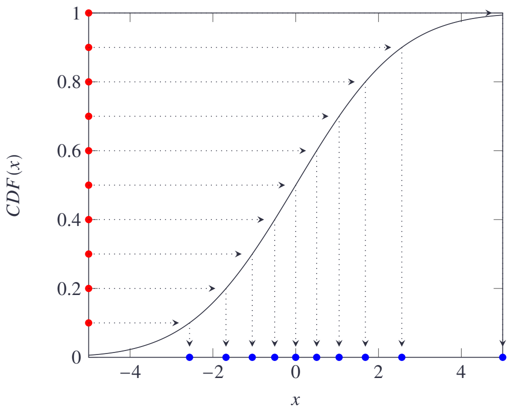

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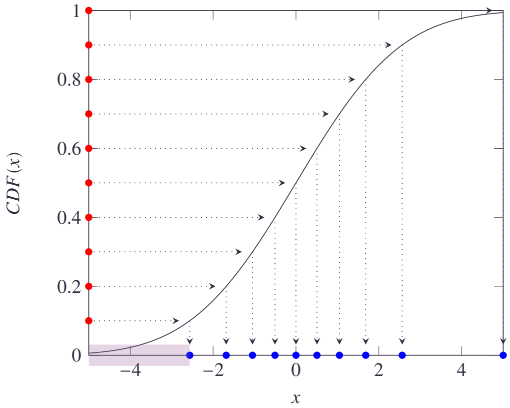

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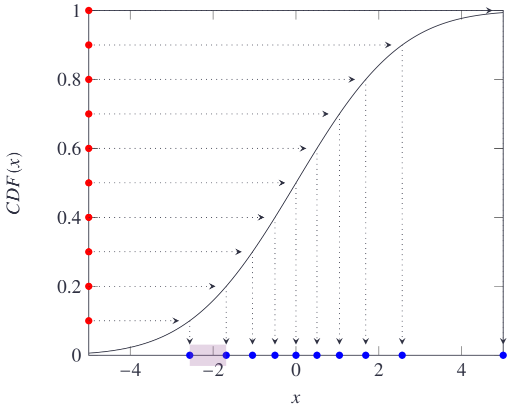

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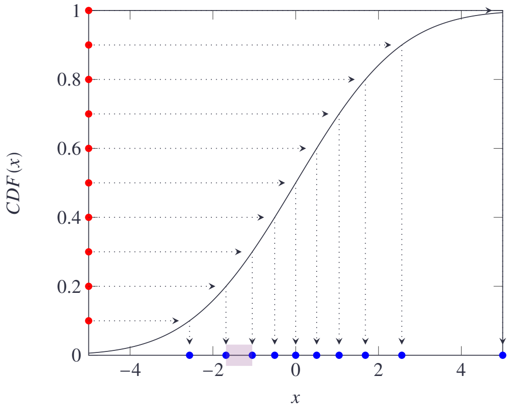


Note: this method assumes we know how to calculate the inverse CDF

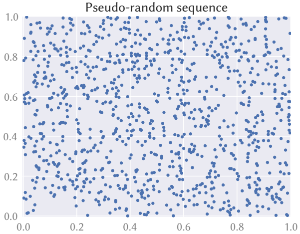
Other sampling techniques

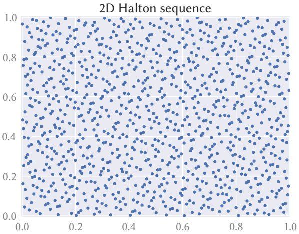
Another random sampling technique you may see in the literature: use of **low-discrepancy sequences** to implement **quasi-Monte Carlo sampling**

- low-discrepancy (or "quasi-random") sequences are constructed deterministically using formulæ
- they fill the input space more quickly than pseudorandom sequences, so lead to faster convergence
- intuition behind these types of sequences: each time you draw a new point it is placed as far away as possible from all points you already have

Low discrepancy sequences

A low discrepancy sequence is a deterministic mathematical sequence that doesn't show clusters and tends to fill space more uniformly than pseudo-random points. (Pseudo-random means as random as you can get when working with a computer.)





Commonly used low discrepancy sequences for Monte Carlo modelling include the Halton sequence and the Sobol' sequence.

More information: see the Jupyter/Python notebook at `risk-engineering.org`

The Saint Petersberg game

- ▷ You flip a coin repeatedly until a tail first appears
  - the pot starts at 1€ and doubles every time a head appears
  - you win whatever is in the pot the first time you throw tails and the game ends
- ▷ For example:
  - T (tail on the first toss): win 1€
  - H T (tail on the second toss): win 2€
  - H H T: win 4€
  - H H H T: win 8€
- ▷ Reminder (see associated slides on *Economic viewpoint on risk transfer*): the expected value of this game is infinite

→ let's estimate the expected value using a Monte Carlo simulation

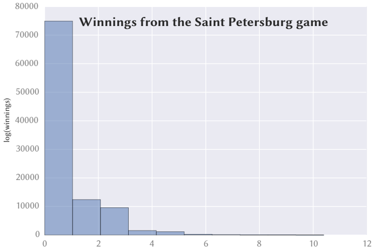
The Saint Petersburg game and limits of Monte Carlo methods

```
import numpy, matplotlib.pyplot as plt

def petersburg():
    payoff = 1
    while numpy.random.uniform() > 0.5:
        payoff *= 2
    return payoff

N = 1_000_000
games = numpy.zeros(N)
for i in range(N):
    games[i] = petersburg()

plt.hist(numpy.log(games), alpha=0.5)
print(games.mean())
12.42241
```



This game illustrates a situation where very unlikely events have an extremely high impact on the mean outcome. Monte Carlo simulation will not allow us to obtain a good estimation of the true (theoretical) expected value.

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- Body mass index chart on slide 10: InvictaHOG from Wikimedia Commons, public domain
- Cat on slide 12: Marina del Castell via `flic.kr/p/otQtCc`, CC BY licence

For more information

- Harvard course on Monte Carlo methods, `harvard.edu/courses/am207/`
- MIT OpenCourseWare notes from the *Numerical computation for mechanical engineers* course
- Article *Principles of Good Practice for Monte Carlo Techniques*, Risk Analysis, 1994, DOI: 10.1111/j.1539-6924.1994.tb00265.x
- Book *The Monte Carlo Simulation Method for System Reliability and Risk Analysis*, Enrico Zio, ISBN: 978-1447145882

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